Chapter 12_*Logarithms*

Solving Exponential and Logarithmic Equations

Here we will make use of what we have learned about exponentials and logarithms to solve equations.

One-to-One Property of Exponential Functions –

If $b^n = b^m$ then $n = m$

We may make use of the above property if we are able to express both sides of the equation in terms of the same base as shown here.

Recall the rule of exponent

$(x^n)^m = x^{nm}$

Solve $5^{3x-2} = 125^{2x}$

$$5^{3x-2} = 125^{2x}$$
$$5^{3x-2} = (5^3)^{2x}$$
$$5^{3x-2} = 5^{6x}$$

Step 1*: Express both sides in terms of the same base.*

Step 2*: Equate the exponents.*

$$3x-2 = 6x$$
$$-2 = 3x$$
$$x = -\frac{2}{3}$$

Step 3*: Solve for the variable.*

A. Solve

$2^x = 8$	$32^{x-2} = 64$	$100^x = 1000^{2x-1}$
$2^x = 8$ $2^x = 2^3$ $x = 3$	$32^{x-2} = 64$ $(2^5)^{x-2} = 2^6$ $2^{5x-10} = 2^6$ $5x-10 = 6$ $5x = 16$ $x = 16/5$	$100^x = 1000^{2x-1}$ $(10^2)^x = (10^3)^{2x-1}$ $10^{2x} = 10^{6x-3}$ $2x = 6x-3$ $-4x = -3$ $x = 3/4$

$81^{-x} = 9^{-x-2}$	$4^{2x-1} = 64$	$36^x = 216^{x-3}$
$81^{-x} = 9^{-x-2}$ $(9^2)^{-x} = 9^{-x-2}$ $9^{-2x} = 9^{-x-2}$ $-2x = -x-2$ $-x = -2$ $x = 2$	$4^{2x-1} = 64$ $4^{2x-1} = 4^3$ $2x-1 = 3$ $2x = 4$ $x = 2$	$36^x = 216^{x-3}$ $(6^2)^x = (6^3)^{x-3}$ $6^{2x} = 6^{3x-9}$ $2x = 3x-9$ $-x = -9$ $x = 9$

It is not always the case that we will be able to express both sides of an equation in terms of the same base. For this reason we will make use of the following property.

One-to-One Property of Logarithmic Functions – For all real b, $b > 0$ and $b \neq 1$

$$\log_b x = \log_b y \quad \text{if and only if} \quad x = y$$

Solve $3^{2x-1} = 10$

$3^{2x-1} = 10$

***Step 1**: Take the common log of both sides.* ⇨ $\log(3^{2x-1}) = \log(10)$

$(2x-1)\log 3 = \log(10)$ ⇦ ***Step 2**: Apply the power rule for logarithms.*

$\frac{(2x-1)\cancel{\log 3}}{\cancel{\log 3}} = \frac{\log(10)}{\log 3}$

***Step 3**: Solve for the variable.* ⇨ $2x - 1 = \frac{1}{\log 3}$

$2x = 1 + \frac{1}{\log 3}$

Exact Answer ⇨ $x = \frac{1}{2} + \frac{1}{2\log 3}$

$x \approx 1.55$ ⇦ *Approximate answer rounded off to the nearest hundredth.*

B. Solve

$7^{3x} = 5$	$e^{-x} = 1$	$e^{x-6} = 150$
$7^{3x} = 5$	$e^{-x} = 1$	$e^{x-6} = 150$
$\log 7^{3x} = \log 5$	$\ln e^{-x} = \ln 1$	$\ln e^{x-6} = \ln 150$
$3x \cdot \log 7 = \log 5$	$-x \cdot \ln e = \ln 1$	$(x-6) \cdot \ln e = \ln 150$
$x = \frac{\log 5}{3\log 7}$	$-x \cdot 1 = 0$	$x - 6 = \ln 150$
$x \approx .2757$	$x = 0$	$x = 6 + \ln 150$
		$x \approx 11.0106$

$2 - 10^x = -28$	$5(0.3)^{x-2} = 30$	$2 + 3^{2x-3} = 10$
$2 - 10^x = -28$	$5(.3)^{x-2} = 30$	$2 + 3^{2x-3} = 10$
$-10^x = -30$	$(.3)^{x-2} = 6$	$3^{2x-3} = 8$
$10^x = 30$	$\log(.3)^{x-2} = \log 6$	$\log 3^{2x-3} = \log 8$
$\log 10^x = \log 30$	$(x-2)\log .3 = \log 6$	$(2x-3)\log 3 = \log 8$
$x \log 10 = \log 30$	$x - 2 = \frac{\log 6}{\log .3}$	$2x - 3 = \frac{\log 8}{\log 3}$
$x = \log 30$	$x = 2 + \frac{\log 6}{\log .3}$	$2x = 3 + \frac{\log 8}{\log 3}$
$x \approx 1.4771$	$x \approx .5118$	$x = \frac{3}{2} + \frac{\log 8}{2 \cdot \log 3}$
		$x \approx 2.4464$

When solving exponential equations and using the above process the rule of thumb is to choose the common logarithm unless the equation involves *e*. We choose these because there is a button for them on the calculator. But certainly we could use any base we wish; this is the basis for the derivation of the change of base formula.

Prove $\log_b x = \dfrac{\log_a x}{\log_a b}$

***Step 1**: Apply the definition of the logarithm.*

$$y = \log_b x$$
$$b^y = x$$

***Step 2**: Take the base a log of both sides.*

$$\log_a b^y = \log_a x$$

***Step 3**: Apply the power rule for logarithms.*

$$y \cdot \log_a b = \log_a x$$

***Step 4**: Solve for y.*

$$y = \frac{\log_a x}{\log_a b}$$

For the last step simply equate the equivalent forms of y above.

$$\text{So} \quad \log_b x = \frac{\log_a x}{\log_a b}$$

We can also use the one-to-one property for logarithms to solve logarithmic equations. If we are given an equation with a logarithm of the same base on both sides we may simply equate the arguments.

$$\log_3 2 + 2\log_3 x = \log_3(7x-3)$$

***Step 1**: Use the rules for logarithms to isolate a log on both sides of the equation.*

$$\log_3 2 + 2 \cdot \log_3 x = \log_3(7x-3)$$
$$\log_3 2 + \log_3 x^2 = \log_3(7x-3)$$
$$\log_3(2x^2) = \log_3(7x-3)$$

***Step 2**: Equate the arguments.*

$$2x^2 = 7x-3$$

***Step 3**: Solve for the variable.*

$$2x^2 - 7x + 3 = 0$$
$$(2x-1)(x-3) = 0$$
$$2x-1=0 \quad \text{OR} \quad x-3=0$$
$$2x = 1$$
$$x = 1/2 \checkmark \qquad x = 3 \checkmark$$
$$\text{Solution set } \{1/2, 3\}$$

***Step 4**: We must check to see if they work.*

Check $x = 1/2$:

$$\log_3 2 + 2\log_3\left(\tfrac{1}{2}\right) \stackrel{?}{=} \log_3\left(7 \cdot \tfrac{1}{2} - 3\right)$$
$$\log_3 2 + \log_3\left(\tfrac{1}{2}\right)^2 \stackrel{?}{=} \log_3\left(\tfrac{1}{2}\right)$$
$$\log_3\left(2 \cdot \tfrac{1}{4}\right) \stackrel{?}{=} \log_3\left(\tfrac{1}{2}\right)$$
$$\log_3\left(\tfrac{1}{2}\right) = \log_3\left(\tfrac{1}{2}\right) \checkmark$$

Check $x = 3$:

$$\log_3 2 + 2\log_3(3) \stackrel{?}{=} \log_3(7 \cdot 3 - 3)$$
$$\log_3 2 + \log_3 3^2 \stackrel{?}{=} \log_3(18)$$
$$\log_3(2 \cdot 9) = \log_3(18) \checkmark$$

Be sure to check to see if the solutions that we obtain solve the original logarithmic equation. In this manual we will put a check mark next to the solution after we determine that it really does solve the equation. This process sometimes results in extraneous solutions so ***we must check our answers***.

C. Solve

$\log_2(3x) = \log_2(2x+7)$

$$\log_2(3x) = \log_2(2x+7)$$
$$3x = 2x+7$$
$$x = 7 \checkmark$$

check: $\log_2(3\cdot 7) \stackrel{?}{=} \log_2(2\cdot 7+7)$

$$\log_2(21) = \log_2(21) \checkmark$$

Solution Set $\{7\}$

$2\ln x = \ln 25$

$$2\ln x = \ln 25$$
$$\ln x^2 = \ln 25$$
$$x^2 = 25$$
$$x = -5 \text{ or } x = 5 \checkmark$$

extraneous

<u>check $x=5$</u>: $2\ln 5 \stackrel{?}{=} \ln 25$

$$\ln 5^2 = \ln 25 \checkmark$$

<u>check $x=-5$</u>: $2\ln(-5) \stackrel{?}{=} \ln 25$

undefined, therefore extraneous

solution set $\{5\}$

$\ln(2x+10) = 2\ln 2$

$$\ln(2x+10) = 2\ln 2$$
$$\ln(2x+10) = \ln 2^2$$
$$2x+10 = 4$$
$$2x = -6$$
$$x = -3 \checkmark$$

Check: $x = -3$

$$\ln(2(-3)+10) \stackrel{?}{=} 2\ln 2$$
$$\ln 4 = \ln 2^2 \checkmark$$

Solution Set $\{-3\}$

$\log_2 5 + \log_2 x = \log_2(x+20)$

$$\log_2 5 + \log_2 x = \log_2(x+20)$$
$$\log_2(5x) = \log_2(x+20)$$
$$5x = x+20$$
$$4x = 20$$
$$x = 5 \checkmark$$

<u>check</u>: $\log_2 5 + \log_2 5 \stackrel{?}{=} \log_2(5+20)$

$$\log_2(5\cdot 5) = \log_2(25)$$
$$\log_2(25) = \log_2(25) \checkmark$$

Solution Set $\{5\}$

$\log(x+1) + \log(x-1) = \log 8$

$$\log(x+1) + \log(x-1) = \log 8$$
$$\log(x+1)(x-1) = \log 8$$
$$\log(x^2-1) = \log 8$$
$$x^2 - 1 = 8$$
$$x^2 = 9$$
$$\cancel{x = -3} \text{ or } x = 3 \checkmark$$

extraneous

Solution Set $\{3\}$

$\log_5 x + \log_5(x-3) = \log_5 10$

$$\log_5 x + \log_5(x-3) = \log_5 10$$
$$\log_5 x(x-3) = \log_5 10$$
$$x^2 - 3x = 10$$
$$x^2 - 3x - 10 = 0$$
$$(x+2)(x-5) = 0$$
$$x+2 = 0 \text{ or } x-5 = 0$$
$$\cancel{x = -2} \qquad x = 5 \checkmark$$

extraneous

solution set $\{5\}$

Of course, equations like these are very special. Most of the problems that we will encounter will not have a logarithm on both sides, as with the next set.

If the answer to our logarithmic equation makes the argument of it negative then it is extraneous. This does not preclude negative answers. We must be sure to check all of our solutions.

$$\log_2 x + \log_2 (x-4) = 5$$

Step 1: Use the rules for logarithms to isolate a log..

$$\log_2 x + \log_2 (x-4) = 5$$
$$\log_2 x(x-4) = 5$$
$$\log_2 (x^2-4x) = 5$$

Step 2: Apply the definition of the logarithm and change it to an exponential equation.

$$x^2-4x = 2^5$$
$$x^2-4x = 32$$
$$x^2-4x-32 = 0$$
$$(x-8)(x+4) = 0$$
$$x-8=0 \quad \text{or} \quad x+4=0$$

Step 3: Solve for the variable.

$$x = 8 \checkmark \qquad x = -4 \text{ extraneous}$$

Solution set $\{8\}$

Step 4: We must check to see if the solutions work.

Check: $x = 8$
$$\log_2 8 + \log_2 (8-4) \stackrel{?}{=} 5$$
$$\log_2 (8 \cdot 4) \stackrel{?}{=} 5$$
$$\log_2 32 = 5 \checkmark$$

Check $x = -4$
$$\log_2 -4 + \log_2 (-4-4) \stackrel{?}{=} 5$$
$$\log_2 -4 + \log_2 (-8) \neq 5$$
undefined

Tip: Not all negative solutions are extraneous! Look at the following problems and we can see that some have negative answers.

D. Solve

$$\log_6 (2x+6) = 2$$

$$\log_6 (2x+6) = 2$$
$$2x+6 = 6^2$$
$$2x+6 = 36$$
$$2x = 30$$
$$x = 15 \checkmark$$

Solution Set $\{15\}$

$$\log(2x+30) = 1$$

$$\log_{10} (2x+30) = 1$$
$$2x+30 = 10^1$$
$$2x = -20$$
$$x = -10 \checkmark$$

Solution Set $\{-10\}$

$$\ln(x^2-4) = 0$$

$$\ln (x^2-4) = 0$$
$$x^2-4 = e^0$$
$$x^2-4 = 1$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

Solution Set $\{-\sqrt{5}, \sqrt{5}\}$

$$\log_{49} (x+1) = \tfrac{1}{2}$$

$$\log_{49} (x+1) = \tfrac{1}{2}$$
$$x+1 = 49^{\frac{1}{2}}$$
$$x+1 = \sqrt{49}$$
$$x+1 = 7$$
$$x = 6 \checkmark$$

Solution Set $\{6\}$

The check mark indicates that we actually plugged the answers in to see that they do indeed solve the original. ***Please do not skip this step***, extraneous solutions occur often.

$\log_2(15x-1)-\log_2(x-1)=4$	$\ln x-\ln(x-1)=1$
$\log_2(15x-1)-\log_2(x-1)=4$ $\log_2\left(\frac{15x-1}{x-1}\right)=4$ $\frac{15x-1}{x-1}=2^4$ $15x-1=16(x-1)$ $15x-1=16x-16$ $-x=-15$ $x=15$ ✓ Solution Set $\{15\}$	$\ln x-\ln(x-1)=1$ $\ln\left(\frac{x}{x-1}\right)=1$ $\frac{x}{x-1}=e^1$ $x=e(x-1)$ $x=ex-e$ $x-ex=-e$ $x(1-e)=-e$ $x=\frac{-e}{1-e}=\frac{e}{e-1}$ ✓ Solution Set $\left\{\frac{e}{e-1}\right\}$

$\log_4 x+\log_4(x-12)=3$	$\log_5 x+\log_5(2x-9)=1$
$\log_4 x+\log_4(x-12)=3$ $\log_4 x(x-12)=3$ $x(x-12)=4^3$ $x^2-12x=64$ $x^2-12x-64=0$ $(x+4)(x-16)=0$ $x+4=0$ OR $x-16=0$ $\cancel{x=-4}$ $x=16$ ✓ extraneous Solution Set $\{16\}$	$\log_5 x+\log_5(2x-9)=1$ $\log_5 x(2x-9)=1$ $x(2x-9)=5^1$ $2x^2-9x-5=0$ $(2x+1)(x-5)=0$ $2x+1=0$ OR $x-5=0$ $2x=-1$ $x=5$ ✓ $\cancel{x=-1/2}$ extraneous Solution Set $\{5\}$

$2\log(11x-12)+1=3$	$\log_6 x=1-\log_6(x-1)$
$2\log(11x-12)+1=3$ $2\log(11x-12)=2$ $\log(11x-12)=1$ $11x-12=10^1$ $11x=22$ $x=2$ ✓ Solution Set $\{2\}$	$\log_6 x=1-\log_6(x-1)$ $\log_6 x+\log_6(x-1)=1$ $\log_6 x(x-1)=1$ $x(x-1)=6^1$ $x^2-x-6=0$ $(x+2)(x-3)=0$ $x+2=0$ OR $x-3=0$ $\cancel{x=-2}$ $x=3$ ✓ extraneous Solution Set $\{3\}$

Be sure to check all solutions to logarithmic equations because some might be extraneous!